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On the equilibrium and stability of a row of point vortices

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The equilibrium and stability of a single row of equidistantly spaced identical point vortices is a classical problem in vortex dynamics, which has been addressed by several investigators in different ways for at least a century. Aspects of the history and the essence of these treatments are traced, stating some in more accessible form, and pointing out interesting and apparently new connections between them. For example, it is shown that the stability problem for vortices in an infinite row and the stability problem for vortices arranged in a regular polygon are solved by the same eigenvalue problem for a certain symmetric matrix. This result also provides a more systematic enumeration of the basic instability modes. The less familiar theory of equilibria of a finite number of vortices situated on a line is also recalled.

1. Introduction

In §156 of his classic text *Hydrodynamics* Lamb (1932) discusses the arrangement and stability of a single row of identical point vortices, elaborating on investigations on single and double rows by von Kármán (1911, 1912; see also von Kármán & Rubach 1912).† Today we view the physical relevance of this model in at least two ways. On one hand, we may think of the row of vortices as a discretized version of a vortex sheet, such as Rosenhead (1931) used in his pioneering numerical 'vortex method' study of vortex sheet roll-up. Then in the continuum limit we should recover from the stability problem for the vortex row the well-known dispersion relation for the inviscid Kelvin–Helmholtz instability. On the other hand, stimulated by the experimental discoveries of coherent structures in a shear layer (cf. Roshko 1976), we can view the single row of vortices as a simplified model of a system of more complex vortices, each represented by a single degree of freedom. In this interpretation the instability modes correspond to observable modes of evolution of the shear layer, and we would expect a 'pairing mode' (Brown & Roshko 1974; Winant & Browand 1974) to predominate.

The problem of a row of vortices has been approached in various ways over the years. Von Kármán discussed the physical problem of an infinite row on the unbounded plane, as reviewed by Lamb (1932). The problem of certain conditionally convergent sums was circumvented by invoking physical arguments. I briefly recapitulate these arguments in §2 since I need them for later developments. (This approach is also reviewed in §7.5 of the recent monograph by Saffman 1992.) A more elegant way of arriving at the results is perhaps to consider the problem of point vortices in a periodic strip, and I show how this is done in §3. The necessary algebraic

† These papers are most easily found in von Kármán (1956).

steps to carry out this analysis were discovered quite independently of the Kármán–Lamb analysis (and without a problem of vortex dynamics in mind) by Calogero & Perelomov (1979, 1978). These results are included in Sec. 15.823 of the enlarged edition of the well-known table by Gradshteyn & Ryzhik (1980). I state the formal results as a theorem of linear algebra in §4.

A different approach to the problem of a row of vortices is to consider a regular polygon of N identical vortices, another equilibrium configuration, and for fixed side length to let the number of vortices increase so that the circle on which they are located gradually expands to infinity and locally looks more and more like a segment of an infinite row. This strategy was used by Havelock (1931) in an important paper, although he was primarily interested in the more complicated problem of the double row, the counterpart of the well-known Kármán 'vortex street'. The vortex polygons were studied independently by Lord Kelvin (Thomson 1878), motivated by experiments of Mayer (1878*a*, *b*) on configurations of floating magnets, and by J. J. Thomson (1883) in his Adams Prize Essay. Various minor inaccuracies in Thomson's (1883) analysis were corrected by Morton (1935) independently of Havelock's (1931) more comprehensive treatment. In §5 I demonstrate that there is a connection between the stability problem for the polygon configurations and the eigenvalue problem stated in §4. In particular, I re-derive Thomson's theorem that a vortex N-gon is stable for $N \leqslant 6$ but unstable for $N \geqslant 8$. Most of the results given in §5 are well known and are reviewed in §7.1 of Saffman's (1992) monograph. However, the connection between the stability problem for the infinite (or periodic) linear arrangement and that for vortex polygons and the key role of the eigenvalue problem of §4 appear to be new.†

A third and considerably less familiar approach to the problem of a linear array of vortices is to consider equilibrium configurations with a finite number of identical vortices situated on a line. These were discovered by Stieltjes in 1885 (see Szegö 1939, §6.7) in the physical context of seeking equilibria of electrostatically interacting line charges, and re-discovered and analysed in considerable detail by Calogero and co-workers in the mid-1970s (cf. Ahmed *et al.* 1979). In particular, it was shown that the linearized stability problem for such a finite line of vortices can be solved analytically. Taking the limit $N \rightarrow \infty$ again leads to the results for the infinite, single row, but some interesting variations are now possible. For example, one finds an equilibrium of identical point vortices with an 'inhomogeneity', a vortex of different strength, embedded. This material is the subject of §§6 and 7. It is based on the papers cited and on some unpublished notes derived from correspondence with F. Calogero in 1979.

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To conclude this Introduction I state the basic equations of motion under study:

$$\frac{\mathrm{d}z_\alpha^*}{\mathrm{d}t} = \frac{1}{2\pi \mathrm{i}} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{1.1}$$

Herein the z_α are the complex positions of the point vortices, $z_\alpha = x_\alpha + \mathrm{i}y_\alpha$, where (x_α, y_α) are the Cartesian coordinates of vortex $\alpha = 1, \ldots, N$. The Γ_α are the circulations or strengths of the vortices. They are constant in time, and for most of the present discussion will be chosen to be identical. Here and in what follows an asterisk denotes complex conjugation, and the prime on the summation sign indicates omission of the singular term $\beta = \alpha$.

† Although I have used this approach in lectures on the subject for a number of years.

2. The von Kármán–Lamb argument

We start from the intuitively clear observation that an infinite row of uniformly spaced, identical point vortices is an equilibrium configuration. The velocity produced at the location of any of the vortices consists of an infinite sequence of equal magnitude but oppositely directed velocities from pairs of vortices at equal distances to the left and right of the vortex in question. Each such pair of contributions cancels.

Formally, we have a time-independent, equilibrium configuration given by

$$z_\alpha^{(0)} = \alpha a, \quad \alpha = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

where a is any chosen spacing. According to (1.1) the stationarity of the configuration amounts to the statement that

$$\sum_\beta{}' \frac{1}{z_\alpha^{(0)} - z_\beta^{(0)}} = \frac{1}{a} \sum_\beta{}' \frac{1}{\alpha - \beta} = 0, \tag{2.2}$$

and this depends on a summability rule for the conditionally convergent sum. The requirement (2.2) is, of course, the only reasonable one from a physical point of view.

Proceeding to a stability analysis we set

$$z_\alpha(t) = z_\alpha^{(0)} + \zeta_\alpha(t), \tag{2.3}$$

substitute in (1.1) and linearize in the perturbations ζ_α. Recalling (2.2) this gives

$$\frac{\mathrm{d}\zeta_\alpha^*}{\mathrm{d}t} = -\frac{\Gamma}{2\pi \mathrm{i}} \sum_\beta{}' \frac{\zeta_\alpha - \zeta_\beta}{(z_\alpha^{(0)} - z_\beta^{(0)})^2} = \frac{\mathrm{i}\Gamma}{2\pi a^2} \sum_\beta{}' \frac{\zeta_\alpha - \zeta_\beta}{(\alpha - \beta)^2}. \tag{2.4}$$

We now notice that if we substitute for ζ_α a plane wave solution, $\exp\{\mathrm{i}ka\alpha\}$ – where the wavenumber k may be chosen to satisfy $|k|a < \pi$ – we obtain on the right-hand side of (2.4)

$$\mathrm{e}^{\mathrm{i}ka\alpha} \frac{\mathrm{i}\Gamma}{2\pi a^2} \sum_\beta{}' \frac{1 - \mathrm{e}^{\mathrm{i}ka(\beta - \alpha)}}{(\beta - \alpha)^2}. \tag{2.5}$$

The sum is clearly independent of α and equals

$$2 \sum_{\beta=1}^{\infty} \frac{1 - \cos(ka\beta)}{\beta^2} = 4 \sum_{\beta=1}^{\infty} \frac{\sin^2(ka\beta/2)}{\beta^2} = \pi |k| a \left(1 - \frac{|k|a}{2\pi}\right). \tag{2.6}$$

The imaginary part vanishes by antisymmetry, and the last equality follows from knowledge of the Fourier series expansion of the function $x(1-x)$ on $0 \leqslant x \leqslant 1$ (cf. Gradshteyn & Ryzhik 1980, equation 1.443.3).

For future use we define

$$\sigma_k = \frac{\Gamma|k|}{2a}\left(1 - \frac{|k|a}{2\pi}\right). \tag{2.7}$$

If we expand a general solution $\zeta_\alpha(t)$ as a Fourier integral over all relevant wavenumbers k,

$$\zeta_\alpha(t) = \int_{-\pi/a}^{\pi/a} \mathrm{d}k\, Z_k(t)\, \mathrm{e}^{-\mathrm{i}ka\alpha}, \tag{2.8}$$

equation (2.4) gives us
$$\dot{Z}_k^* = \mathrm{i}\sigma_{-k} Z_{-k} = \mathrm{i}\sigma_k Z_{-k}, \tag{2.9}$$
from which
$$\ddot{Z}_k = -\mathrm{i}\sigma_k \dot{Z}_{-k}^* = \sigma_k^2 Z_k. \tag{2.10}$$

It follows that every wavelike perturbation, except the pure displacement mode $k = 0$ (or $k = \pm 2\pi/a$), can lead to exponential growth at a rate σ_k. The most unstable modes occur for $k = \pm\pi/a$. Since these modes have a wavelength $\lambda = 2\pi/|k| = 2a$, they can be captured even at finite amplitude by considering the dynamics of a periodically continued system with just two vortices in the 'basic cell', a theme that we shall elaborate on in §4. The wavelength $2a$ is the shortest wave that can be accommodated by the periodic steady state under consideration.

Hence, von Kármán (1912) concluded that the single, infinite row of uniformly spaced, identical vortices is unconditionally unstable, i.e. unstable to a wavelike perturbation of any wavenumber that the state can support. The growth rate, σ_λ, of a perturbation wave of wavelength λ is given by (2.7):

$$\sigma_\lambda = \frac{\pi\Gamma}{a\lambda}\left(1-\frac{a}{\lambda}\right). \tag{2.11}$$

The argument was given as a preparation for the more complicated analysis of vortex streets. A slightly more elaborate version may be found in von Kármán & Rubach (1912), in Lamb (1932, §156) and in Saffman (1992, §7.5).

In the continuum limit, $\lambda \gg a$, we may replace Γ/a by the velocity jump ΔU across the vortex sheet, and the term in parentheses in (2.11) by 1. The resulting formula for the growth rate,

$$\sigma_\lambda = \frac{\pi \Delta U}{\lambda}, \tag{2.12}$$

then reproduces the classical formula for the inviscid Kelvin–Helmholtz instability (cf. Landau & Lifshitz 1987, (29.8), who refer to this flow as a 'tangential discontinuity').

3. Vortices in a periodic strip

I want now to embark on something that may at first sight appear unrelated, but that will quickly be linked with the developments in the preceding section. This is the notion of a periodically continued system of point vortices.

If in (1.1) we assume that a collection of N point vortices is repeated periodically along the x-axis, we must augment the mutual interactions given in that equation to include the effect of the 'periodic images' of each vortex. Thus, assume that in addition to the N vortices with positions $z_\alpha, \alpha = 1, \ldots, N$, we have for each an infinite row of uniformly spaced periodic images located at

$$z_\alpha^{(m)} = z_\alpha + mL, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{3.1}$$

where L is the length of the period. The 'basic' vortices at $z_\alpha, \alpha = 1, \ldots, N$, can always be thought of as those representatives of the infinite periodic family that at any instant are in the 'basic strip', $0 \leqslant x \leqslant L$, but, of course, any representative from each periodic family will do.

Substitution in (1.1) now gives

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N}{}' \Gamma_\beta \sum_{m=-\infty}^{\infty} \frac{1}{z_\alpha - (z_\beta + mL)}. \tag{3.2}$$

The contributions from the periodic images of vortex α itself cancel in pairs just as in the argument for the equilibrium configuration in §2. The sum over m in (3.2) may be rewritten as

$$\sum_{m=-\infty}^{\infty} \frac{1}{z_\alpha - (z_\beta + mL)} = \frac{1}{z_\alpha - z_\beta} + 2\sum_{m=1}^{\infty} \frac{z_\alpha - z_\beta}{(z_\alpha - z_\beta)^2 - (mL)^2}. \tag{3.3}$$

Using the partial fraction expansion of cot we recognize this as $(\pi/L)\cot\{(\pi/L)(z_\alpha - z_\beta)\}$. Thus, we have the counterpart of (1.1) for a periodically continued system with N vortices in the 'basic strip':

$$\frac{\mathrm{d}z_\alpha^*}{\mathrm{d}t} = \frac{1}{2L\mathrm{i}} \sum_{\beta=1}^{N}{}' \Gamma_\beta \cos\left\{\frac{\pi}{L}(z_\alpha - z_\beta)\right\}. \tag{3.4}$$

Let us immediately substitute in this equation the result from §2 that (2.1) is a solution. Indeed, if we set $a = L/N$, we can capture this state in the periodic system with N vortices in a strip of width L. This gives us the identity

$$\sum_{\beta=1}^{N}{}' \cot\left\{\frac{\pi}{N}(\alpha - \beta)\right\} = 0. \tag{3.5}$$

Since this holds for every vortex, the sum is independent of α, as one can also easily verify directly, and we have the simpler form of the result:

$$\sum_{\mu=1}^{N-1} \cot\left(\frac{\mu\pi}{N}\right) = 0, \tag{3.6}$$

a trigonometric identity valid for all N.

Let me make a few remarks in closing this section. The first is that (3.4) could also be found by using the Routh–Lin Green's function theory for point-vortex motion in bounded domains (see Lin 1943). The second is that the dynamical system (3.4) is again Hamiltonian, and retains both components of linear impulse as integrals. This seems first to have been noticed by Birkhoff & Fisher (1959). The invariant Hamiltonian is

$$H = -\frac{1}{2\pi} \sum_{1 \leqslant \alpha < \beta \leqslant N} \Gamma_\alpha \Gamma_\beta \log\left|\sin\left\{\frac{\pi}{L}(z_\alpha - z_\beta)\right\}\right|. \tag{3.7}$$

The system (3.4) is, thus, integrable for $N = 1, 2$ and arbitrary vortex strengths. For $N = 3$ it is integrable if the three vortices have zero net circulation, but probably not otherwise (Aref 1985). Situations where such motions are relevant, I suggest, include some of the three-vortex-per-cell wakes observed experimentally by Williamson & Roshko (1988), in particular the regime they call 'P+S' in their paper. This theme is currently being pursued independently and will be reported elsewhere. Integration of the problem of three interacting vortices with vanishing net circulation on the infinite plane was considered a few years ago by Rott (1989) and Aref (1989). The corresponding problem in the periodic strip appears to be considerably more involved.

4. An eigenvalue problem

We may now give a slightly different flavour to the developments in §2 by noting that any wave with a wavelength λ such that a/λ is rational, say $a/\lambda = p/q$, where p, q are natural numbers with $2p \leqslant q$ (since $\lambda \geqslant 2a$), must repeat after every q vortices. Hence, we must be able to capture such a wave by considering the system of q vortices periodically continued. In other words, if we perform the linearized stability analysis on such a periodic system, we should find unstable wavenumbers that are precisely the same as those uncovered in §2.

Let us write out this observation explicitly. We have now a steady state

$$z_\alpha^{(0)} = \alpha a, \quad \alpha = 1, \ldots, q, \tag{4.1}$$

in place of (2.1). The superscript (0) may be read also as $m = 0$ in the context of (3.1), and then $L = qa$. The dynamics in question is that of (3.4). Hence, when we perturb similarly to (2.3),

$$z_\alpha(t) = z_\alpha^{(0)} + \zeta_\alpha(t), \tag{4.2}$$

where again $\alpha = 1, \ldots, q$, we obtain in place of (2.4)

$$\frac{d\zeta_\alpha^*}{dt} = \frac{i\pi\Gamma}{2L^2} \sum_{\beta=1}^{q}{}' \frac{\zeta_\alpha - \zeta_\beta}{\sin^2\{(\pi/q)(\alpha-\beta)\}}, \tag{4.3}$$

with the strip width $L = qa$.

Note that the coefficient of ζ_α on the right-hand side is independent of α:

$$\sum_{\gamma=1}^{q}{}' \frac{1}{\sin^2\{(\pi/q)(\alpha-\gamma)\}} = \sum_{\gamma=1}^{q}{}' \frac{1}{\sin^2\{(\pi/q)\gamma\}}. \tag{4.4}$$

According to the arguments given connecting the periodic system to the infinite system considered in §2, the perturbation $\zeta_\alpha = \exp\{i2\pi p\alpha/q\}$ reproduces itself multiplied by the factor

$$\frac{i\pi\Gamma}{q^2a^2}|p|(q-|p|), \tag{4.5}$$

which is just $i\sigma_k$ from (2.7) with k rewritten as $(2\pi/a)(p/q)$. (We have let p assume both positive and negative values here.)

This result is equivalent to the statement that the $q \times q$ real symmetric matrix,

$$A_{\alpha\beta} = \delta_{\alpha\beta} \sum_{\gamma=1}^{q}{}' \frac{1}{\sin^2\{(\pi/q)\gamma\}} - (1-\delta_{\alpha\beta}) \frac{1}{\sin^2\{(\pi/q)(\alpha-\beta)\}}, \tag{4.6}$$

which appears on the right-hand side of (4.3), has the eigenvalues

$$a_p^{(q)} = 2p(q-p), \quad p = 0, 1, \ldots, [q/2], \tag{4.7}$$

and that associated with each of these there are two eigenvectors, $\phi_\alpha^{(p)} = \cos(2\pi p\alpha/q)$ and $\psi_\alpha^{(p)} = \sin(2\pi p\alpha/q), \alpha = 1, \ldots, q$. (When q is even, we have only $\phi_\alpha^{(p)}$ for $p = q/2$. For both even and odd q we have only $\phi_\alpha^{(p)}$ for $p = 0$.) These statements clearly solve the eigenvalue problem for the matrix $A_{\alpha\beta}$ in (4.6) completely. They were stated in this form by Calogero & Perelomov (1978, 1979).

We can now turn the entire argument around. Let us consider from the outset the periodically continued system with q vortices in a basic strip of width $L = qa$. We note the equilibrium corresponding to a uniformly spaced row of vortices, and address the linearized stability problem for this configuration. Thus, we are led to (4.3). We now use the result that the eigenvalue problem for the coefficient matrix appearing herein can be solved as just indicated. The growth rates and eigen-perturbations have been given above. Specifically, for a system with q vortices we obtain growth rates

$$\sigma_p^{(q)} = \frac{\pi\Gamma}{2L^2} a_p^{(q)} = \frac{\pi\Gamma}{a^2} \frac{p}{q}\left(1 - \frac{p}{q}\right), \quad p = 0, 1, \ldots, [q/2]. \tag{4.8}$$

The final result agrees, of course, with the final result of §2, but the modes are now enumerated in a systematic way according to the value of q, the number of independent vortices in the periodic system.

The fastest growing mode, $p = q/2$, is realized for every strip that has an even number of vortices. It is first realized for $q = 2$, and since this problem is integrable, can be followed to finite amplitude. In particular, the complex 'vector' from vortex 1 to vortex 2, $z_1 - z_2$, satisfies $|\sin\{(\pi/L)(z_1 - z_2)\}| = \text{const.}$, or

$$\cosh\left\{\frac{2\pi}{L}(y_1 - y_2)\right\} - \cos\left\{\frac{2\pi}{L}(x_1 - x_2)\right\} = 2. \tag{4.9}$$

This equation for the relative position of the two vortices is considerably simpler than the expression given by Saffman & Baker (1979, equation (3.4)).†

There are modes with particular symmetries for larger q that may also be followed to finite amplitude. These results have languished in unpublished theses by Tengara (1981) and Blomberg (1984). Figure 5 of Aref (1985) reproduces results of Blomberg (1984) that show possible trajectories of the vortices for the cases $q = 2$, 3 and 4. Particularly interesting are the separatrix motions in which the point vortices exchange places along the row (albeit in infinite time). Thus, using the $q = 2$ mode, a row of vortices ...ABCDEF... can become ...BADCFE.... This is the point-vortex counterpart of the 'pairing mode' (no merging of vortices can occur for point vortices). Using the $q = 3$ mode a row ...ABCDEF... can become ...BACEDF.... Finally, for $q = 4$ we can change ...ABCDEFGH... into ...CBADGFEH.... These results suggest that by exciting different modes with different periodicities one may be able to achieve interesting mixing patterns in a real shear layer that has rolled up into discrete vortices. I believe that some of these modes are intimately related to the patterns observed in excited shear layers by Ho and coworkers (see Ho & Huang 1982; Ho & Huerre 1984) more than a decade ago. For different q the modes have different ranges of 'spatial reach' along the layer. The $q = 2$ mode involves interactions of nearest neighbours. For $q = 3$ we have interactions of second-nearest neighbours, etc. Of course, finite-area vortices of the same sign will typically merge at some point during their mutual orbit, and this must be kept in mind when interpreting the point-vortex motions in the context of real flows.

There is an important postscript to these analyses of stability of finite vortex systems. When we impose the various perturbations, such as (4.2) and (5.4) below, it is important to distinguish perturbations that leave the integrals of motion invariant (to linear order) from those that do not. For example, invariances of the linear impulse implies that a perturbation such as ζ_α in (4.2) satisfy

$$\sum_{\alpha=1}^{q} \Gamma_\alpha \zeta_\alpha = 0. \tag{4.10}$$

(I have written the condition for the general case of different strengths.) For identical vortices this condition is, in fact, satisfied for the wavelike perturbations $\zeta_\alpha = \exp\{\mathrm{i}2\pi p\alpha/q\}$ so long as $p \neq 0$. Angular impulse is not conserved for the periodic strip, but conservation of kinetic energy, i.e. invariance of the Hamiltonian under perturbation, leads to the following result: Consider the vortex equations of motion written in Hamiltonian form

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}; \quad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \tag{4.11}$$

† This is corrected on p. 133 of Saffman (1992). J. T. Stuart has kindly pointed out that the result (4.9) was first obtained by Rosenhead (1931); see also Stuart (1986).

where for the periodic strip H is given by (3.7). Then a perturbation $\delta\zeta_\alpha = \delta x_\alpha + \mathrm{i}\delta y_\alpha$ implies the following change in the value of H:

$$\delta H = \sum_{\alpha=1}^{q} \left(\frac{\partial H}{\partial x_\alpha} \delta x_\alpha + \frac{\partial H}{\partial y_\alpha} \delta y_\alpha \right) = \sum_{\alpha=1}^{q} \Gamma_\alpha (\dot{x}_\alpha \delta y_\alpha - \dot{y}_\alpha \delta x_\alpha). \tag{4.12}$$

This formula shows two things. First, if the unperturbed state is steady, $\dot{x}_\alpha = \dot{y}_\alpha = 0$, then there is no change in H. Second, if the unperturbed state consists of uniform rotation of each vortex, such that $\dot{z}_\alpha = \mathrm{i}\Omega z_\alpha$, or $\dot{x}_\alpha = -\Omega y_\alpha, \dot{y}_\alpha = \Omega x_\alpha$, then

$$\delta H = -\Omega \sum_{\alpha=1}^{q} \Gamma_\alpha (x_\alpha \delta x_\alpha + y_\alpha \delta y_\alpha) = -\tfrac{1}{2}\Omega \delta I, \tag{4.13}$$

where I is the angular impulse of the vortex system,

$$I = \sum_{\alpha=1}^{q} \Gamma_\alpha (x_\alpha^2 + y_\alpha^2). \tag{4.14}$$

Hence, for the configurations we are considering it suffices that perturbations leave linear and angular impulse invariant (the latter will automatically ensure invariance of the Hamiltonian). If the unperturbed state is steady, one need only check invariance of linear impulse under perturbation.

5. Vortex polygons

We return to the unbounded plane and in (1.1) set all $\Gamma_\alpha = \Gamma$ and make the Ansatz $z_\alpha(t) = R\exp[\mathrm{i}\{\Omega t + \alpha 2\pi/N\}], \alpha = 1, \ldots, N$. The identical vortices are now situated at the vertices of a regular N-gon that rotates rigidly with angular frequency Ω. From (1.1) we obtain the equation

$$\frac{2\pi\Omega R^2}{\Gamma} = \sum_{\beta=1}^{N}{}' \{1 - \mathrm{e}^{\mathrm{i}(\beta-\alpha)2\pi/N}\}^{-1}. \tag{5.1}$$

The sum on the right is independent of α and equals

$$\sum_{\mu=1}^{N-1} (1 - \mathrm{e}^{\mathrm{i}\mu 2\pi/N})^{-1} = \frac{1}{2} \sum_{\mu=1}^{N-1} \left\{ 1 + \mathrm{i}\cot\left(\frac{\mu\pi}{N}\right) \right\} = \tfrac{1}{2}(N-1). \tag{5.2}$$

The first transformation follows from elementary trigonometry, the second from (3.6), that we gave a physical interpretation in §3.

We have thus established a connection between the angular frequency of rotation, Ω, and the radius of the circle through the vortices, R:

$$\Omega R^2 = \frac{\Gamma}{4\pi}(N-1). \tag{5.3}$$

We designate the solution to (1.1) that we have just found by $z_\alpha^{(0)}(t)$, and consider the perturbed solution

$$z_\alpha(t) = z_\alpha^{(0)}(t)\{1 + \eta_\alpha(t)\}. \tag{5.4}$$

Substitute this into (1.1), and linearize in the perturbations $\eta_\alpha(t)$. After subtraction of the equation of motion for the unperturbed solution there results the following system:

$$\left(\frac{\mathrm{d}\eta_\alpha^*}{\mathrm{d}t} - \mathrm{i}\Omega\eta_\alpha^* \right) z_\alpha^{(0)*} = -\frac{\Gamma}{2\pi\mathrm{i}} \sum_{\beta=1}^{N}{}' \frac{z_\alpha^{(0)}\eta_\alpha - z_\beta^{(0)}\eta_\beta}{(z_\alpha^{(0)} - z_\beta^{(0)})^2}. \tag{5.5}$$

The denominators in the sum on the right-hand side may be written as

$$-4R^2 \exp\{\mathrm{i}2\pi(\alpha+\beta)/N\} \sin^2\{\pi(\alpha-\beta)/N\}. \tag{5.6}$$

Then (5.5) becomes

$$\frac{\mathrm{d}\eta_\alpha^*}{\mathrm{d}t} - \mathrm{i}\Omega\eta_\alpha^* = \frac{\Gamma}{8\pi \mathrm{i} R^2} \sum_{\beta=1}^{N}{}' \frac{\exp\{\mathrm{i}(2\pi/N)(\alpha-\beta)\}\,\eta_\alpha - \eta_\beta}{\sin^2\{(\pi/N)(\alpha-\beta)\}}. \tag{5.7}$$

The coefficient of η_α on the right-hand side contains the sum

$$\sum_{\beta=1}^{N}{}' \frac{\exp\{\mathrm{i}(2\pi/N)(\alpha-\beta)\}}{\sin^2\{(\pi/N)(\alpha-\beta)\}} = \sum_{\beta=1}^{N}{}' \frac{\cos\{(2\pi/N)(\alpha-\beta)\} + \mathrm{i}\sin\{(2\pi/N)(\alpha-\beta)\}}{\sin^2\{(\pi/N)(\alpha-\beta)\}}, \tag{5.8}$$

the imaginary part of which vanishes according to (3.6). The real part is

$$\sum_{\beta=1}^{N}{}' \frac{\cos\{(2\pi/N)(\alpha-\beta)\}}{\sin^2\{(\pi/N)(\alpha-\beta)\}} = \sum_{\beta=1}^{N}{}' \frac{1}{\sin^2\{(\pi/N)(\alpha-\beta)\}} - 2(N-1). \tag{5.9}$$

The sum is independent of α, as we saw in (4.4). We note the reappearance in (5.7) of the matrix $A_{\alpha\beta}$ from (4.6).

In view of (5.8), (5.9) and (5.3), (5.7) now becomes

$$\frac{\mathrm{d}\eta_\alpha^*}{\mathrm{d}t} - \mathrm{i}\Omega\eta_\alpha^* - \mathrm{i}\Omega\eta_\alpha = \frac{-\mathrm{i}\Gamma}{8\pi R^2} \sum_{\beta=1}^{N} A_{\alpha\beta}\eta_\beta, \tag{5.10}$$

where we know the eigenvalues and eigenvectors of the matrix on the right-hand side.

Let us divide (5.10) through by $\Gamma/8\pi R^2$, using (5.3) to relate this factor to Ω, and let us write η_α as a column vector $\boldsymbol{\eta}$ with real and imaginary parts $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}''$, respectively. We also write $\boldsymbol{A}$ for the matrix $A_{\alpha\beta}$. With these changes (5.10) becomes

$$\dot{\boldsymbol{\eta}}' - \mathrm{i}\dot{\boldsymbol{\eta}}'' - \mathrm{i}4(N-1)\boldsymbol{\eta}' = -\mathrm{i}\boldsymbol{A}\boldsymbol{\eta}' + \boldsymbol{A}\boldsymbol{\eta}'' \tag{5.11}$$

(the overdot denotes a derivative with respect to time scaled by $\Gamma/8\pi R^2$). Separating into real and imaginary parts we obtain

$$\dot{\boldsymbol{\eta}}' = \boldsymbol{A}\boldsymbol{\eta}'', \quad \dot{\boldsymbol{\eta}}'' + 4(N-1)\boldsymbol{\eta}' = \boldsymbol{A}\boldsymbol{\eta}'. \tag{5.12a, b}$$

Thus,

$$\ddot{\boldsymbol{\eta}}' = \boldsymbol{A}\dot{\boldsymbol{\eta}}'' = \boldsymbol{A}\{\boldsymbol{A} - 4(N-1)\,\mathbf{1}\}\boldsymbol{\eta}', \tag{5.13}$$

where $\mathbf{1}$ is the unit matrix. It is clear from this equation that the growth rates for unstable modes of the polygon are given by

$$\sigma_p^{(N\text{-gon})} = \pm\frac{\Gamma}{8\pi R^2}[a_p^{(N)}\{a_p^{(N)} - 4(N-1)\}]^{1/2}, \quad p = 0, 1, \ldots, [N/2], \tag{5.14a}$$

where the $a_p^{(N)}$ are given by (4.7), i.e.

$$\sigma_p^{(N\text{-gon})} = \pm\frac{\Gamma}{4\pi R^2}[p(N-p)\{p(N-p) - 2(N-1)\}]^{1/2}, \quad p = 0, 1, \ldots, [N/2]. \tag{5.14b}$$

This is the result given by Havelock (1931) who used polar coordinates to perform the stability analysis. It was rediscovered by Dritschel (1985) who considered the corresponding states of finite-area vortices. Havelock's derivation is reviewed in the monograph by Saffman (1992, §7.1).

We briefly considered the allowed perturbations in the sense of the discussion at the

N	$2(N-1)$	$p(N-p)$
2	2	1
3	4	2
4	6	3, 4
5	8	4, 6
6	10	5, 8, 9
7	12	6, 10, 12
8	14	7, 12, 15, 16
9	16	8, 14, 18, 20

TABLE 1. Values of N, $(2N-1)$ and $p(N-p)$ for $p = 1, 2, \ldots, [N/2]$

end of the previous section. The eigenmodes are again waves, and all p listed are allowed. Considering the angular impulse, the condition for invariance of I to linear order in the perturbation is that the sum of the real parts of the perturbations η_α vanishes. The condition of invariance of linear impulse is that the sum of $\eta_\alpha \exp\{i2\pi\alpha/N\}$ vanishes. Since η_α depends on α as $\exp\{i2\pi p\alpha/N\}$, we see that linear impulse is always left invariant (because η_α is included as a factor of $z_\alpha^{(0)}(t)$ in equation (5.4)), and angular impulse is left invariant except for the $p = 0$ mode, which corresponds to a perturbation to a polygon of slightly different size. This is always a neutrally stable mode. Thus, all values of the growth rate listed in (5.14*b*) except $p = 0$ are to be considered for the problem of stability of the polygon configuration when the values of the linear impulse, angular impulse and energy (Hamiltonian) are left invariant.

Table 1 shows the values of N, $2(N-1)$ and $p(N-p)$ for the appropriate range of p. We see that $p(N-p) < 2(N-1)$ for all p when $2 \leqslant N \leqslant 6$. Hence, the growth-rates in (5.14*b*) are pure imaginary, i.e., $\sigma_p^{(N\text{-gon})} = i\omega$, where ω is a frequency of oscillation, and there is no instability. For $N \geqslant 8$ at least one mode satisfies $p(N-p) > 2(N-1)$ and we have instability. For $N = 7$ the modes with $p = 3, 4$ have a zero growth-rate and are neutrally stable in linear theory. The transition from stable polygons to instability as N crosses 7 is known as Thomson's theorem, since it was first enunciated in Thomson's (1883) Adams Prize Essay. The proof given here and the connections revealed between this problem and the problem of the infinite row appear to be new.

Lord Kelvin (Thomson 1878) noted that for $N = 3$ the frequency of oscillation, ω, coincides with the frequency of rotation of the equilateral triangle of vortices, Ω. Indeed, this is true for the $p = 1$ modes for any value of N, i.e. (5.14*b*) with $p = 1$ reproduces (5.3) for Ω. Khazin (1976) presents an argument that the N-gon with $N \leqslant 6$ is not only stable in linear theory but is Lyapunov stable as well. Mertz (1978) considers the effect of adding a central vortex.

In the limit $N \to \infty$, $R \to \infty$, with the ratio $2\pi R/N = a$ held fixed, the angular frequency $\Omega \to 0$, and the ring tends (locally) to a row with spacing a. The growth rate $\sigma_p^{(N\text{-gon})}$ tends to (2.11) when we remember that the mode p corresponds to a wave of wavelength $\lambda = 2\pi R/p$.

6. Collinear configurations

We conclude our study of the single row by exploring in this section and the next a family of finite point-vortex equilibria in which the vortices are all on a line. If the vortices are identical, this line must rotate rigidly. Let us first determine the positions of the vortices along the line in this case.

We consider the Ansatz $z_\alpha(t) = x_\alpha \exp(\mathrm{i}\Omega t), \alpha = 1, \ldots, N$, where the x_α are certain numbers to be determined. Substitution in (1.1) gives a system of algebraic equations,

$$\frac{2\pi\Omega}{\Gamma} x_\alpha = \sum_{\beta=1}^{N}{}' \frac{1}{x_\alpha - x_\beta}, \quad \alpha = 1, \ldots, N, \tag{6.1}$$

a nonlinear eigenvalue problem for determining both the frequency of rotation and the positions along the line.

Equation (6.1) may be solved by using a 'generating polynomial' of the vortex configuration,

$$P(x) = (x - x_1) \ldots (x - x_N). \tag{6.2}$$

This function has the important property that its logarithmic derivative is proportional to the fluid velocity at x:

$$P'(x) = P(x) \sum_{\alpha=1}^{N} \frac{1}{x - x_\alpha}. \tag{6.3}$$

Thus,

$$P''(x) = P(x) \left\{ \sum_{\alpha,\beta=1}^{N} \frac{1}{(x - x_\alpha)(x - x_\beta)} - \sum_{\alpha=1}^{N} \frac{1}{(x - x_\alpha)^2} \right\} = P(x) \sum_{\alpha \neq \beta} \frac{1}{(x - x_\alpha)(x - x_\beta)}. \tag{6.4}$$

In this sum one decomposes the summand in partial fractions. A sum such as the one on the right-hand side of (6.1) is then obtained. Substitution of the left-hand side of (6.1) and simplification of the result yields the following ODE for $P(x)$:

$$\frac{\Gamma}{2\pi\Omega} P'' - 2xP' + 2NP = 0. \tag{6.5}$$

Except for the factor multiplying P'', which can be eliminated by a rescaling of the independent variable x, this is the differential equation of the Hermite polynomial of degree N. In particular, one has the result obtained by Stieltjes (see Szegö 1939, §6.7) in a different physical context that the positions of the vortices along the rigidly rotating line are given by

$$x_\alpha = x_\alpha^{(N)} \left(\frac{\Gamma}{2\pi\Omega} \right)^{1/2}, \quad \alpha = 1, \ldots, N, \tag{6.6}$$

where the $x_\alpha^{(N)}$ are the zeros of the Nth Hermite polynomial.

We may notice in passing that a closely related analysis and use of the identity that the derivative of H_N is $2NH_{N-1}$ shows that the stagnation points of the flow induced by this linear array of identical vortices are located at the positions of the zeros of the $(N-1)$th Hermite polynomial (scaled again by $\{\Gamma/2\pi\Omega\}^{1/2}$ as in (6.6)).

An interesting by-product of this development – much as we found (3.6) in our earlier analysis – arises by multiplying (6.1) by x_α and summing on α. The right-hand side can easily be shown to equal $N(N-1)/2$. On the left-hand side we obtain from (6.6) the sum of $(x_\alpha^{(N)})^2$. Hence, we derive a 'sum rule' for the roots of the Nth Hermite polynomial:

$$\sum_{\alpha=1}^{N} (x_\alpha^{(N)})^2 = \tfrac{1}{2}N(N-1). \tag{6.7}$$

It is worth discussing a simple generalization of the above procedure for finding collinear equilibria. If N is odd, $N = 2n+1$, $x = 0$ is a root, and the symmetrically positioned vortices rotate about a central vortex. Equilibria exist where this vortex does not have the same strength as the others. For example, if its strength is $p\Gamma$, the nonlinear eigenvalue problem (6.1) is replaced by

$$\frac{2\pi\Omega}{\Gamma}x_\alpha = \frac{p}{x_\alpha} + \sum_{\beta=1}^{N-1}{}' \frac{1}{x_\alpha - x_\beta}, \quad \alpha = 1, \ldots, N-1, \tag{6.8}$$

where we have numbered the vortices such that the stationary vortex at the origin is vortex N. Since the $2n$ identical vortices are symmetrically placed pairwise, the equation expressing the stationarity of vortex N at the origin is trivial, and (6.8) can be rewritten as

$$\frac{2\pi\Omega}{\Gamma}x_\alpha = \frac{p+\frac{1}{2}}{x_\alpha} + \sum_{\beta=1}^{n}{}' \frac{2x_\alpha}{x_\alpha^2 - x_\beta^2}, \quad \alpha = 1, \ldots, n. \tag{6.9}$$

We are now summing over only half the vortices, the symmetrically placed partner of each having been taken into account in the sum. The first term on the right is the contribution from the vortex at the origin, and the vortex at $-x_\alpha$. We may note from (6.9), the counterpart of (5.3), that

$$\frac{2\pi\Omega}{\Gamma} = (p+\tfrac{1}{2})\frac{1}{n}\sum_{\alpha=1}^{n}\frac{1}{x_\alpha^2}. \tag{6.10}$$

Using a generating function similar to (6.2), but with a factor x^p to handle the vortex at the origin, Ahmed *et al.* (1979) have shown that if $p > -\frac{1}{2}$, (6.10) has a unique solution in which the $(2\pi\Omega/\Gamma)\,x_\alpha^2$ are the roots of the generalized Laguerre polynomial $L_n^{(p-1/2)}$. For $p = 1$ we should return to the case of $2n+1$ identical vortices, and, indeed, $H_{2n+1}(x)$ is proportional to $L_n^{(1/2)}(x^2)$. Similarly, for $p = 0$ we should return to the case of $2n$ identical vortices, and, indeed, $H_{2n}(x)$ is proportional to $L_n^{(-1/2)}(x^2)$.

For $p \leqslant -\frac{1}{2}$ there is a solution for $N = 3$ but not for larger N. (For $p = -\frac{1}{2}$ this solution has $\Omega = 0$ as follows from (6.10)). This is easily seen by considering (6.9) for the smallest and largest values of x, i.e. for $\alpha = 1$ and $\alpha = n$. Dividing these two equations by x_1 and x_n, respectively, and subtracting gives

$$0 = (p+\tfrac{1}{2})\left(\frac{1}{x_1^2}-\frac{1}{x_n^2}\right) + 2\sum_{\beta=2}^{n-1}\left\{\frac{1}{x_1^2-x_\beta^2} - \frac{1}{x_n^2-x_\beta^2}\right\} + 2\left\{\frac{1}{x_1^2-x_n^2} - \frac{1}{x_n^2-x_1^2}\right\}, \tag{6.11a}$$

or

$$0 = (p+\tfrac{1}{2})\left(\frac{1}{x_1^2}-\frac{1}{x_n^2}\right) + 2\sum_{\beta=2}^{n-1}\frac{x_n^2-x_1^2}{(x_1^2-x_\beta^2)(x_n^2-x_\beta^2)} + \frac{4}{x_1^2-x_n^2}. \tag{6.11b}$$

In this equation, for $p \leqslant -\frac{1}{2}$ and $n \geqslant 2$, the first and second terms are $\leqslant 0$, the third is < 0. Hence, no solution of (6.9) exists for $N > 3$ when $p \leqslant -\frac{1}{2}$.

Of particular interest for our present discussion are the limiting forms of these results when $N \to \infty$. It is well known that the zeros of the Hermite polynomials become uniformly spaced in this limit. For large N, $x_\alpha^{(N)}$ varies as $\alpha\pi/(2N)^{1/2}$. As the configuration converges to one with a fixed spacing, a, between the vortices, the angular frequency diminishes according to $\Omega = \pi\Gamma/4Na^2$ from (6.6).

A very interesting result is obtained for the case of a vortex of strength $p\Gamma$ ($p > -\frac{1}{2}$) at the origin when we let $N \to \infty$. This now becomes a single row with an 'imperfection', a vortex that is either a bit stronger or weaker than the rest. The x_α now tend to the zeros of the Bessel function of order $p-\frac{1}{2}$. Indeed, the zeros $j_{p-1/2,\alpha}$ of $J_{p-1/2}(x)$ satisfy the identity (Ahmed *et al.* 1979)

$$\frac{p+\frac{1}{2}}{j_{p-1/2,\alpha}} + \sum_{\beta=1}^{\infty}{}' \frac{2j_{p-1/2,\alpha}}{j_{p-1/2,\alpha}^2 - j_{p-1/2,\beta}^2} = 0. \tag{6.12}$$

Thus, except for an undetermined scale factor, the positions within a steady, infinite, single row of point vortices with an 'imperfection' that is p times stronger than the others are given by the zeros of the Bessel function of order $p-\frac{1}{2}$. For $p = 0$ or 1 we recover the single row of von Kármán and Lamb (§2) as a reflection of the familiar results that the zeros of $J_{\pm 1/2}$ are uniformly spaced (with spacing π).

7. Linear stability of collinear configurations

Remarkably, the linear stability problem for the state given by (6.6) can be carried out analytically for arbitrary N. The required algebraic results are given by Bruschi (1979). We proceed as in our earlier analyses. Set

$$z_\alpha(t) = (x_\alpha + \zeta_\alpha(t))\exp(\mathrm{i}\Omega t), \tag{7.1}$$

where the x_α are determined according to (6.6). Invariance of the linear impulse now implies that the ζ_α must sum to zero. Invariance of the angular impulse implies the orthogonality relation

$$\sum_{\alpha=1}^{N} x_\alpha^{(N)}\zeta'_\alpha = 0 \tag{7.2}$$

where the prime again indicates the real part.

Substitute (7.1) into (1.1) and linearize. This leads after a straightforward calculation to

$$\frac{\mathrm{d}\zeta^*_\alpha}{\mathrm{d}t} - \mathrm{i}\Omega\zeta^*_\alpha = \mathrm{i}\Omega \sum_{\beta=1}^{N}{}' \frac{\zeta_\alpha - \zeta_\beta}{(x_\alpha^{(N)} - x_\beta^{(N)})^2}. \tag{7.3}$$

Let us write ζ_α as a column vector $\boldsymbol{\zeta}$ with real and imaginary parts $\boldsymbol{\zeta}'$ and $\boldsymbol{\zeta}''$, respectively. We also write $\boldsymbol{A}$ for the matrix $A_{\alpha\beta}$ with elements

$$A_{\alpha\alpha} = \sum_{\beta=1}^{N}{}' \frac{1}{(x_\alpha^{(N)} - x_\beta^{(N)})^2}; \quad A_{\alpha\beta} = \frac{-1}{(x_\alpha^{(N)} - x_\beta^{(N)})^2} \quad \text{for} \quad \alpha \neq \beta. \tag{7.4a, b}$$

Using this notation (7.3) becomes

$$\dot{\boldsymbol{\zeta}}' = -\Omega(\boldsymbol{A}-1)\boldsymbol{\zeta}'', \quad \dot{\boldsymbol{\zeta}}'' = -\Omega(\boldsymbol{A}+1)\boldsymbol{\zeta}'. \tag{7.5a, b}$$

The eigenvalue problem for matrix $\boldsymbol{A}$ has been solved completely for general N (Bruschi 1979). The eigenvalues are $N-m, m = 1, \ldots, N$. The corresponding (unnormalized) eigenvectors are $\boldsymbol{v}^{(m)}$ with components

$$v_\alpha^{(m)} = H_{m-1}(x_\alpha^{(N)})/H_{N-1}(x_\alpha^{(N)}), \quad \alpha = 1, \ldots, N. \tag{7.6}$$

From (7.5) we have

$$\ddot{\boldsymbol{\zeta}} = \Omega^2(\boldsymbol{A}^2 - 1)\boldsymbol{\zeta}. \tag{7.7}$$

In view of (7.7) the linearized eigenmodes have growth rates

$$\sigma_m = \Omega[(N-m)^2 - 1]^{1/2}, \quad m = 1, \ldots, N. \tag{7.8}$$

The modes $m = N-1$ and $m = N$ are neutrally stable. These modes correspond to the eigenvectors $v_\alpha^{(N-1)} = x_\alpha^{(N)}/(N-1)$, i.e. a scaling of the configuration, and $v_\alpha^{(N)} = 1$, i.e. a finite displacement of the entire configuration, respectively. (The result $v_\alpha^{(N-1)} = x_\alpha^{(N)}/(N-1)$ follows from the recursion formula $H_N(x) = 2xH_{N-1}(x) - 2(N-1)H_{N-2}(x)$ for Hermite polynomials.) Invariance of linear and angular impulse under perturbation rules out these modes. The remaining modes, $m = 1, \ldots, N-2$, are all orthogonal to $v^{(N-1)}$ and $v^{(N)}$ and therefore respect the invariance of linear and angular impulse (and thus of the Hamiltonian) according to (4.10)–(4.14) and (7.2). These modes are present for $N \geqslant 3$ and are all unstable. The most unstable mode occurs for $m = 1$. It has growth rate $\sigma_1 = \Omega[N(N-2)]^{1/2}$ and the components of the corresponding eigen-perturbation, $v_\alpha^{(1)} = 1/H_{N-1}(x_\alpha^{(N)}), \alpha = 1, \ldots, N$, alternate in sign. In the limit $N \to \infty$, we obtain $\sigma_1 \approx \Omega N \approx \pi\Gamma/4a^2$, which is just (2.11) with $\lambda = 2a$.

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